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Thermal Entanglement of the Two-Qubit Heisenberg Spin Chain Coupled to a Single-Mode Cavity Field

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Abstract Thermal entanglement of a two-qubit Heisenberg spin chain coupled to a single-mode cavity field is investigated. It is found that (1) thermal entanglement without the rotating-wave approximation (RWA) is explicitly smaller than that obtained with the RWA, which means that the counter-rotating terms have a large impact on thermal entanglement, therefore they cannot be neglected; (2) the case ($\omega \ll \Omega$) is more beneficial for enhancing thermal entanglement than the resonance case ($\omega = \Omega$), the near-resonant case ($\omega \approx \Omega$) and the case ($\omega \gg \Omega$); (3) for thermal entanglement, there is a competition process between the exchange coupling J (the direct-coupling between the two two-level atoms) and the coupling constant g (which deduces the indirect effect between the two two-level atoms); the critical value of g increases with the spin coupling strength J.

Keywords Thermal entanglement · Heisenberg spin chain · Single-mode cavity field · Concurrence

1 Introduction

From the quantum solution of Deutsch's problem [1, 2], we can see that quantum entanglement is essential for the physical realization of quantum computing which is established on the basis of quantum parallelism. That is why quantum entanglement is viewed as a uniquely quantum mechanical resource [3]. Considering that the actual qubits always interact with a certain physical environment and eventually reach a thermal equilibrium state, we should concern about the entanglement of this thermal equilibrium state between the qubits (thermal entanglement). Since the problem of thermal entanglement [4, 5] was firstly introduced

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and studied in Heisenberg spin chains, researchers have been making a large ongoing effort in recent years for better understanding the properties of quantum entanglement, and found that thermal entanglement can be effectively enhanced by improving the anisotropic parameter of spin chain system [6–11] or inhomogeneity of external magnetic field [8–10, 12, 13]. In addition, thermal entanglement of the two two-level atoms interacting with a cavity field [14–16] is also investigated; however, the direct interaction (*e.g.* dipole-dipole interaction) between the two two-level atoms is neglected for the atoms are far enough apart.

On the other hand, for quantum information processing, various candidate physical realizations schemes (Optical photon, Optical cavity quantum electrodynamics, Ion traps, Nuclear magnetic resonance, Quantum dots, Super-conducting qubits and Heisenberg spin chain) contain their own advantages and disadvantages [3], so any individual physical system may not be a potentially good candidate for handling quantum information. Just as M.A. Nielsen pointed, "Maybe there is a great deal of merit to pursuing hybrid designs which attempt to marry the best features of two or more existing technologies [3]." Based on this idea and considering that the spin-spin spacing (nano-scale) in the spin chain is far less than the width of the micro-cavity (micron-scale), thus we have attempted to place a Heisenberg spin chain into a micro-cavity. This is just the main motivation of this paper. Furthermore, considering the ideal boundary condition of the micro-cavity, the field in the micro-cavity is approximate to a single-mode field, and the main interaction between the two-level atom and the cavity field is the electric dipolar interaction. Through the theoretical studying of this physical model, we may better understand and control thermal entanglement between the two two-level atoms by adjusting parameters of the cavity field.

This paper is organized as follows. In the next section we introduce the Hamiltonian of the two-qubit Heisenberg spin chain coupled to a single-mode cavity field model and explain the computation procedures of thermal entanglement between the two two-level atoms. In Sect. 3, we present the detailed results and give the corresponding discussions. Finally, the conclusion of our study is given in Sect. 4.

2 Physical Model and Calculation Procedures

2.1 The Hamiltonian of Our Physical Model

The Hamiltonian of the two-qubit Heisenberg spin chain coupled to a single-mode cavity field is given by

$$\hat{H} = \frac{\omega}{2}\hat{\sigma}_1^z + \frac{\omega}{2}\hat{\sigma}_2^z + \Omega\hat{a}^\dagger\hat{a} + J\hat{\vec{\sigma}}_1 \cdot \hat{\vec{\sigma}}_2 + g\hat{\sigma}_1^x(\hat{a}^\dagger + \hat{a}) + g\hat{\sigma}_2^x(\hat{a}^\dagger + \hat{a}) \tag{1}$$

where ω is the transition frequency of the two-level atom, Ω is the frequency of the single-mode cavity field, J is the Heisenberg exchange coupling between the two two-level atoms, and g is the electric dipole coupling constant of the two-level atom and the single-mode cavity field. $\hat{\vec{\sigma}}$ is the famous Pauli vector ($\hat{\sigma}^z$ and $\hat{\sigma}^x$ are its components along the z and x directions respectively). $\hat{a}^\dagger$ ($\hat{a}$) is the creation (annihilation) operator of the single-mode cavity field. We are working in the units so that ω, Ω, J, g are dimensionless.

2.2 The Calculation Procedures of Thermal Entanglement Between the Two Two-Level Atoms

The density matrix of the total system at thermal equilibrium (temperature T) is $\hat{\rho}_{total}(T) = \exp(-\hat{H}/T)/Z$, where $\hat{H}$ is the Hamiltonian of the total system, the Boltzmann constant is set as 1 and $Z = Tr[\exp(-H/T)]$ is the partition function. Considering that, at thermal equilibrium (finite temperature T), the single-mode cavity field has a lower probability distributions at high excited states than at ground state or low excited states. So in the actual numerical evaluation, we can truncate the dimension (the single-mode cavity field's Fock space) into a finite dimensional vector space ($Dim \times Dim$). Actually, we firstly carry out the calculation of the entanglement degree at a high temperature T in the cases of 50×50 and 100×100 respectively ; then we contrast the results of the entanglement degree. Because the relative error is enough little (e.g. $< 10^{-3}$), we are sure that the Fock space 100×100 is enough precise for the calculation of entanglement degree below the former temperature T.

Then we can immediately get the reduced density matrix $\rho(T)$ of the subsystem (the two two-level atoms) by using a partial trace operation to $\rho_{total}(T)$, which traces out the degrees of the single-mode cavity's freedom.

$$\rho(T) = Tr_{cavity}[\rho_{total}(T)] \tag{2}$$

After obtaining the subsystem's reduced density matrix $\rho(T)$, thermal entanglement between subsystem (the two two-level atoms) can be measured by the concurrence C [17, 18], which is defined as $C = \max\{0, 2\max(\lambda_i) - \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$, where λ_i are the square roots of the eigenvalues of the matrix $R(T) = \rho(T)S\rho^*(T)S$, $S = \hat{\sigma}_1^y \otimes \hat{\sigma}_2^y$, and $*$ stands for the complex conjugate. The concurrence is a very good entanglement degree measurement for the two two-level atom system and applies to both pure and mixed state.

3 Results and Discussions

From the foregoing calculation procedures of thermal entanglement, we can see that the concurrence C is a function of all kinds of parameters in the Hamiltonian $\hat{H}$ and the temperature T. Then we calculate the concurrence C as the function of the coupling constant g and the temperature T, and give three groups of results in the following figures.

In the electric dipole interaction between a two-level atom and a single-mode cavity field, the coupling terms containing $\hat{\sigma}^+\hat{a}^\dagger$ and $\hat{\sigma}\hat{a}$ are generally dropped as a fairly good approximation (RWA). But in Fig. 1 (the concurrence $C(g, T)$ is plotted in the cases of non-RWA and the RWA when $\omega = 1$, $\Omega = 1$, $J = 0$), we can clearly see that the peak value of the concurrence C without RWA (Fig. 1(a), 0.0731062) is explicitly smaller than that obtained with RWA (Fig. 1(b), 0.499945), which means that the counter-rotating terms have a large impact on thermal entanglement (the virtual process occurs and plays an important role), therefore they cannot be neglected. In addition, we find that, both in Fig. 1(a) and Fig. 1(b), the value of the concurrence C of thermal equilibrium state at $T = 0$ is 1, which is not related with the coupling constant g.

Then in Fig. 2, under the non-rotating-wave approximation (Non-RWA), we plot the concurrence $C(g, T)$ in the cases of $\Omega = 0.1$ and $\Omega = 9$ when $\omega = 1$, $J = 0$. All three curves besides Fig. 1(a) show that the two two-level atoms without the direct coupling (for $J = 0$) can become entangled due to interacting with the single-mode cavity field respectively. Therefore the single-mode cavity field vividly plays the role of a bridge in the generation of entanglement between the two separated atoms. Moreover the peak values of the concurrence C in Fig. 1(a), Fig. 2(a) and Fig. 2(b) are 0.0731062, 0.00179347 and 0.452970 respectively. By comparing these peak values, we can find that the case ($\omega \ll \Omega$) is more beneficial for enhancing thermal entanglement than the resonance case ($\omega = \Omega$) and the case ($\omega \gg \Omega$). This result could be explained as follows from the perspective of energy: for

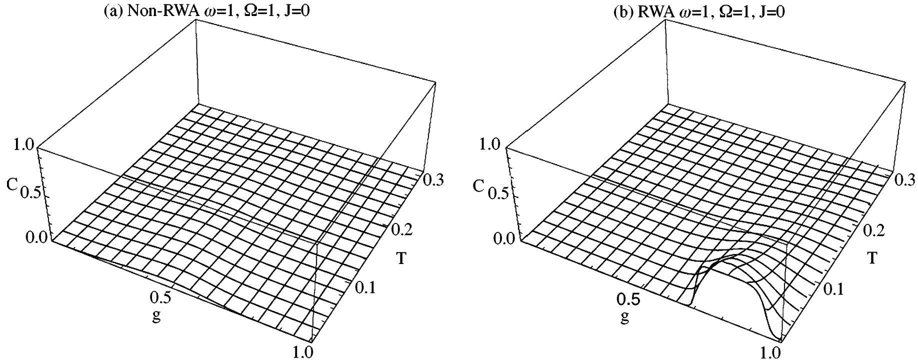

Fig. 1 The concurrence C as a function of the coupling constant g and the temperature T with $\omega = 1$, $\Omega = 1$, $J = 0$, in the cases of (**a**) Non-RWA, (**b**) RWA

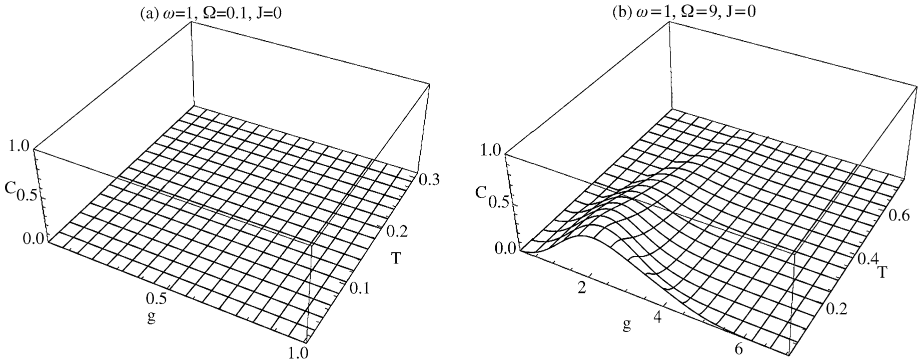

Fig. 2 The concurrence C as a function of the coupling constant g and the temperature T with $\omega = 1$, $J = 0$, in the cases of (**a**) $\Omega = 0.1$, (**b**) $\Omega = 9$

a fixed ω, the larger the parameter Ω (energy of photon) is, the greater the probability of single-photon event (one high-energy photon is absorbed by the two two-level atoms for energy level transitions) is, and then the better the effectiveness of suppression of spontaneous radiation dissipation of the two two-level atoms is. Hence the large detuning case ($\omega \gg \Omega$) is the most opportune case for obtaining an attainable large entanglement degree at thermal equilibrium. We also find that, both in Fig. 1(a) and Fig. 1(b), the value of the concurrence C of thermal equilibrium state at $T = 0$ is 1, which is not related to the coupling strength g.

In addition, the concurrence $C(g, T)$ in the cases of $J = 0.6$ and $J = 1.6$ is given in Fig. 3 under the condition of a large detuning (for $\Omega = 9$). The values of the critical coupling constant g for which the concurrence C vanishes in Fig. 2(a) and Fig. 2(b) are 2.16 and 3.77 respectively. It can be found that the critical coupling constant g has a shift in its increasing direction with the increasing spin coupling strength J. Thus, we can say that for thermal entanglement, there is a competition process between the coupling constant g and the exchange coupling J (the direct-coupling between the two two-level atoms). In fact, the electric dipole coupling constant g (of the two-level atom and the single-mode cavity field) can deduce an indirect effect between the two two-level atoms. Through the unitary S-transformation [19] ($e^{\hat{S}}\hat{H}e^{-\hat{S}}$, where $\hat{S} = \frac{g}{\Omega}(\hat{\sigma}_1^x + \hat{\sigma}_2^x)(\hat{a}^\dagger - \hat{a})$), and by using the famous

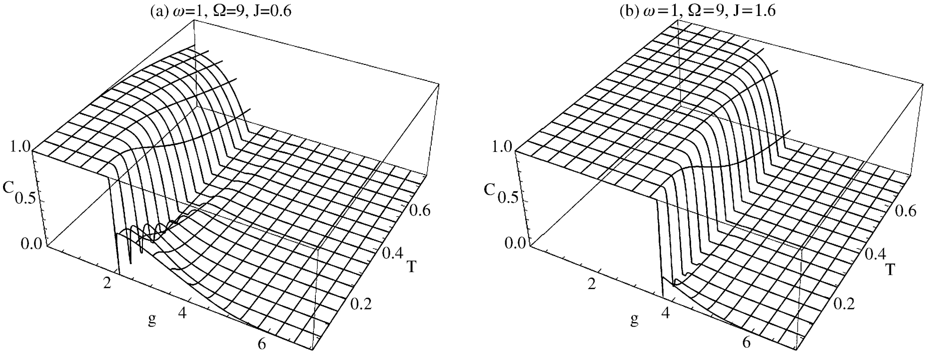

Fig. 3 The concurrence C as a function of the coupling constant g and the temperature T with $\omega = 1$, $\Omega = 9$, in the cases of (**a**) $J = 0.6$, (**b**) $J = 1.6$

Campbell-Baker-Hausdorff relation,

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + [\hat{A}, \hat{B}] + \frac{1}{2}[\hat{A}, [\hat{A}, \hat{B}]] + \cdots \tag{3}$$

we can obtain the Hamiltonian $\hat{H}'$ in a new quantum mechanical representation:

$$\hat{H}' = e^{\hat{S}}\hat{H}e^{-\hat{S}} = \frac{\omega}{2}\sum_{j=1}^{2}(\hat{\sigma}_j^z \cosh(2\alpha) - i\hat{\sigma}_j^y \sinh(2\alpha)) + \Omega\hat{a}^\dagger\hat{a} - \frac{2g^2}{\Omega^2}$$
$$+ \left(J - \frac{g^2}{\Omega}\right)\hat{\sigma}_1^x\hat{\sigma}_2^x + J(\hat{\sigma}_1^y\hat{\sigma}_2^y + \hat{\sigma}_1^z\hat{\sigma}_2^z) \tag{4}$$

where $\alpha = \frac{g}{\Omega}(\hat{a}^\dagger + \hat{a})$. In the new representation by the S-transformation, the spin-boson coupling which does not commute with the qubit Hamiltonian disappears. We can find that transformation renormalizes the single-mode cavity's interaction Hamiltonian into the qubit-qubit interaction. In the Hamiltonian $\hat{H}'$, the term $-\frac{g^2}{\Omega}\hat{\sigma}_1^x\hat{\sigma}_2^x$ (similar to a direct-coupling) is just the indirect effect which arise from the coupling constant g. That is to say, due to the qubit-cavity coupling, the isotropic Hamiltonian of Heisenberg spin chain is changed to the anisotropic Hamiltonian. For $J > 0$, the absolute value of $(J - \frac{g^2}{\Omega})$ firstly decreases to zero and then increase as the coupling constant g; correspondingly, the zero points of the concurrence C in the curves just reflect the competition between the direct interaction (exchange coupling J) and the indirect interaction (coupling constant g).

4 Conclusion

In summary, we have investigated the thermal entanglement of the two-qubit Heisenberg spin chain coupled to a single-mode cavity field. We find that the counter-rotating terms within the electric dipolar interaction have a large inhibited impact on thermal entanglement, and then cannot be neglected. Thereupon, the thermal entanglement without RWA are examined in the cases of the exchange coupling $J = 0$ and $J \neq 0$. It is found that the thermal entanglement can be effectively enhanced under large detuning conditions ($\omega \ll \Omega$) for the case $J = 0$, and there is a competition process between the exchange coupling J (for the case $J > 0$) and the coupling constant g.

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